HOW TO MODEL POST-CRACKING TORSIONAL STIFFNESS AND WHY IT MATTERS IN DESIGN

Edvard P.G. Bruun, Allan Kuan, and Evan C. Bentz

Synopsis: Post-cracking stiffness is an important parameter in determining the proper distribution of forces in the analysis of statically indeterminate reinforced concrete structures. While the ACI 318-19 code specifies typical values to use in modelling flexural cracking, the same guidance is not available when calculating post-cracking torsional stiffness. This paper presents a summary of the academic literature on the topic as the basis for developing a novel stiffness-based design procedure, which is then implemented in the design case study of a spandrel beam supporting a cantilevered roof slab. This example demonstrates a situation where a specific torsional stiffness is required to satisfy serviceability requirements. The design method is general and, therefore, applicable to any situation where an accurate measure of torsional stiffness or moment redistribution is required – this removes the need to iteratively model and design to capture post-cracking effects in structural members.

Keywords: reinforced concrete; torsion; post-cracking stiffness; serviceability; spandrel beam; redistribution

ACI Young Professional Member **Edvard P.G. Bruun**, P.Eng, is a PhD student in the Civil and Environmental Engineering Department at Princeton. He received his bachelor and master's degree from the University of Toronto, after which he worked as a structural engineer at Arup. He is the 2019 recipient of the ACI's Daniel W. Falconer Memorial Fellowship for graduate studies. His research interests include reinforced concrete mechanics, finite element modelling, generative design, and robotic fabrication.

ACI Student Member **Allan Kuan** is a Ph.D. candidate at the University of Toronto. He received his bachelor's degree in Engineering Science from the University of Toronto in 2015. His research interests include developing design procedures and nonlinear analysis tools for reinforced and prestressed concrete members subject to shear and torsion.

ACI Member **Evan C. Bentz**, FACI, is a Professor of civil engineering at the University of Toronto. He received his bachelor's degree from the University of Waterloo, Waterloo, ON, Canada, in 1994, and his PhD from the University of Toronto in 2000. He is former Chair of ACI Committee 365, Service Life Modeling, and is a member of Joint ACI-ASCE Committee 445, Shear and Torsion.

INTRODUCTION

Modern structural engineers have access to a variety of resources to guide them in the strength design of reinforced and prestressed concrete members subjected to torsion. However, rational design procedures for torsion are still a relatively new development compared to design provisions for axial load and flexure – current torsional code provisions are mainly centered around the goal of achieving a safe strength design, compared to the more comprehensive provisions for flexural design. While this focus is understandable given the importance of life safety, there is little information in both academic literature and design codes which consider torsional effects in the context of designing for serviceability.

A thorough code such as the ACI 318-19[1] will contain several guidelines on the topic of flexural cracked stiffness, which are used by engineers to design for strength (i.e. to model the redistribution of forces) or serviceability (i.e. to model deflections). Since the true post-cracking stiffness of a member is a function of its reinforcement, certain rules of thumb regarding cracked stiffness are utilized by designers to capture these effects at the outset of a design to avoid excessive iteration – tables 6.6.3.1.1(a) and (b) in ACI 318-19 are commonly utilized to represent cracked stiffnesses in an elastic analysis.

On the other hand, similar guidelines for post-cracking torsional stiffness are not available, and the knowledge on how to properly model torsion, as opposed to other section resultants, is still not as readily available. To illustrate this, before proceeding with this paper perform a simple experiment and ask a practicing structural engineer the following:

> *For a cracked beam, without performing a calculation, state a reasonable percentage of the uncracked a) flexural and b) torsional stiffness to use in modelling and design.*

We predict that the flexural case would prompt some standard answer taken from the cracked stiffness tables (e.g. 35% I_g), while the torsional case would lead to significantly more ambiguity. A common answer to the question of torsional stiffness might even be that it simply does not matter, but this is only a valid principal for the strength design of statically determinate structures where the torsion is necessary to maintain equilibrium and cannot be redistributed (i.e. the magnitude of resulting torsion is independent of stiffness). In statically indeterminate structures where redistribution based on relative stiffnesses occurs, or in situations where deflection and rotation criteria are important, representing the torsional stiffness of the member correctly is still important.

The lack of explicit code guidance when considering torsion in serviceability situations was the catalyst for this paper, which will attempt to demystify the topic so that the knowledge is more readily available for future practical applications. The paper will begin with an overview of torsional design and a review of the existing literature, following with an explanation of a new stiffness-based design procedure, and concluding with an application of this procedure in a design case study of a spandrel beam whose design is governed by serviceability requirements.

REVIEW OF LITERATURE AND TORSONAL DESIGN PROCEDURES

Significant research on the topic of torsion in reinforced and prestressed concrete was first aggregated and published in SP-018[2], Torsion of Structural Concrete, but it was only in 1971 that detailed procedures for torsion were included in the ACI 318 design code[3]. The following decade saw a flurry of activity on the subject, such as SP-035[4], Analysis of Structural Systems for Torsion, and seminal work by Collins[5,6], Mitchell[6], Lampert[5,7] and Hsu[8], which are effectively the foundation for the current design approaches for torsion.

Conceptually, one of the most important principles to understand when designing for torsion in reinforced concrete structures is the distinction between equilibrium and compatibility torsion, discussed by Collins and Lampert[5], which refers to how the torsional stiffness changes after cracking. Paraphrasing the commentary to clause 22.7.3 in ACI 318-19, these principles can be summarized as follows:

a) ***Equilibrium Torsion:*** *The torsional moment cannot be reduced by redistribution of internal forces… [if] the torsional moment is required for the structure to be in equilibrium.*

b) ***Compatibility Torsion:*** *The torsional moment can be reduced by redistribution of internal forces after cracking if the torsion results from the member twisting to maintain compatibility of deformations.*

Torsional Stiffness in Strength Design

Designing for equilibrium torsion, which typically occurs in statically determinate structures, is relatively straight-forward because there is only one possible path for the load to take. The design moments are simply a function of the geometry and external global loading and are independent of the member stiffnesses. If this load path happens to result in torsion, this member will always experience the same magnitude of moment independent of its level of cracking. In contrast, compatibility torsion is the result of an internal action in a statically indeterminate structure, whereby a member undergoes a twist as it deforms together with the adjacent structure. The relative member stiffnesses become important in determining the distribution of flexural and torsional moments.

The design procedure for an indeterminate structure is simplified, since the true cracked torsional stiffness is rarely known at the outset of a design. One common approach is to reduce the torsional stiffness of a member to approximately 0 to reflect the large drop in torsional stiffness, as observed in experiments on beams[5] and shells[9,10]. Although this is a simplification, as the member will intuitively have some non-zero stiffness, this approach removes the need to consider relative stiffnesses and their influence on the torsional load path. In fact, so long as the surrounding members are adequately designed for the additional flexural moments resulting from redistribution, this approach has been shown to result in safe ultimate strength designs[5]. Modelling with the reduced torsional stiffnesses has the added benefit of 'releasing' the compatibility portion of the resultant torsion; only the remaining equilibrium portion remains to be designed for. This is a valid approach, assuming that the applied torsions are above that required to crack the member, and that the member is adequately detailed for ductility and the resulting rotation.

The commentary to clause 6.3.1.1 in ACI 318-19 states the conditions for considering torsional stiffness in analysis:

> *"Two conditions determine whether it is necessary to consider torsional stiffness in the analysis of a given structure: 1) the relative magnitude of the torsional and flexural stiffness; and 2) whether torsion is required for equilibrium of the structure (equilibrium torsion) or is due to members twisting to maintain deformation compatibility (compatibility)"*

Torsional Stiffness in Serviceability Design

To neglect torsional stiffness is a trade-off between simplicity in design and realistic structural behavior – when proper redistribution is accounted for, this compromise is acceptable from the standpoint of safety. But to apply this same approach when designing for serviceability criteria (for both equilibrium and compatibility torsion situations) is not always appropriate as it will over-estimate the magnitude of rotations and deflections. While it is uncommon that the twisting of a member will have a significant impact on the overall deflections of a structure, there are situations where precise tolerances necessitate that the torsional stiffness of a member be known as accurately as possible. As an added benefit, this stiffness information can always be used to better inform the analysis of an indeterminate structure, which is an update to the zero-stiffness assumption. This is also consistent with a need to consider torsional stiffness as per the commentary excerpt in the previous section. The importance of using proper stiffnesses is further alluded to in the commentary to clause 6.3.1.1, which was expanded in ACI 318-19 to include the following statement:

> *"Separate analyses with different stiffness assumptions may be performed for different objectives such as to check serviceability and strength criteria or to bound the demand on elements where stiffness assumptions are critical"*

This is also stated in clause R6.6.3.2.2, which is generally applied to modelling for serviceability conditions:

> *"Analyses of defections, vibrations, and building periods are needed at various service (unfactored) load levels to determine the performance of the structure in service. The moments of inertia of the structural members in the service load analyses should be representative of the degree of cracking at the various service load levels investigated."*

Yet even though torsional stiffness is mentioned as an important factor, there is no explicit guidance on what values of torsional stiffness to use for either serviceability checks or in the analysis of indeterminate structures.

Methods for Calculating Torsional Stiffness of Reinforced Concrete Members

For a member which primarily resists torsional moments by means of circulating shear stresses, the relationship between the torque and the twist of the member is:

$$T = GK \cdot \psi \tag{1}$$

Where T is the torque, G is the shear modulus of the material, K is the St. Venant's torsion constant and ψ is the twist of the member in units of angle per unit length. The product GK is typically referred to as the torsional stiffness.

Prior to cracking, the torsional stiffness can be calculated using elastic theory as developed by St. Venant. For example, the uncracked stiffness GK_g is calculated using the following formula for members with a rectangular cross section:

$$GK_g = G \cdot \beta b^3 h \tag{2}$$

Where b and h are the short and long sides of the rectangle respectively and β is a coefficient related to the aspect ratio of the member. Values of β are shown in Table 1.

Table 1: Typical values of β

h/b	1.00	1.25	1.50	1.75	2.00	2.50	3.00	4.00	5.00	10.00	∞
β	0.141	0.172	0.196	0.214	0.229	0.249	0.263	0.281	0.291	0.312	0.333

Following cracking of a member loaded in pure torsion, the circulating shear stresses are now carried by spiraling fields of diagonal compression in the concrete which are equilibrated by tensile stresses in the longitudinal and transverse reinforcement. Figure 1 shows the typical torque-twist response of a reinforced concrete member subjected to pure torsion. After cracking occurs, there is a transition region which is approximately linear until either direction of steel yields. The precise behavior in this transition region is not well defined and is highly dependent on when cracking takes place and the presence of accompanying axial load, shear and moment. For this reason, the post-cracking stiffness used in this paper, GK_{cr}, will be taken as the torque when yielding of either direction of reinforcement occurs divided by the corresponding twist. The benefit of using this secant approach is that GK_{cr} is independent of the level of torsion present, making it conceptually consistent with the use of first-order elastic analyses which are typical in engineering practice. It also overestimates the twist under service loads and overestimates the resulting design torque at ultimate limit states, making it a simple yet conservative approach for both strength and serviceability calculations.

It should be noted that although there are other definitions of the post-cracking torsional stiffness which have been used by other researchers (such as by Hsu[8] and Tavio and Teng[11]), the secant stiffness approach suggested here is the simplest and most practical approach for designers trying to understand how the torsional stiffness of their members affects the displacements and member forces in their analysis models.

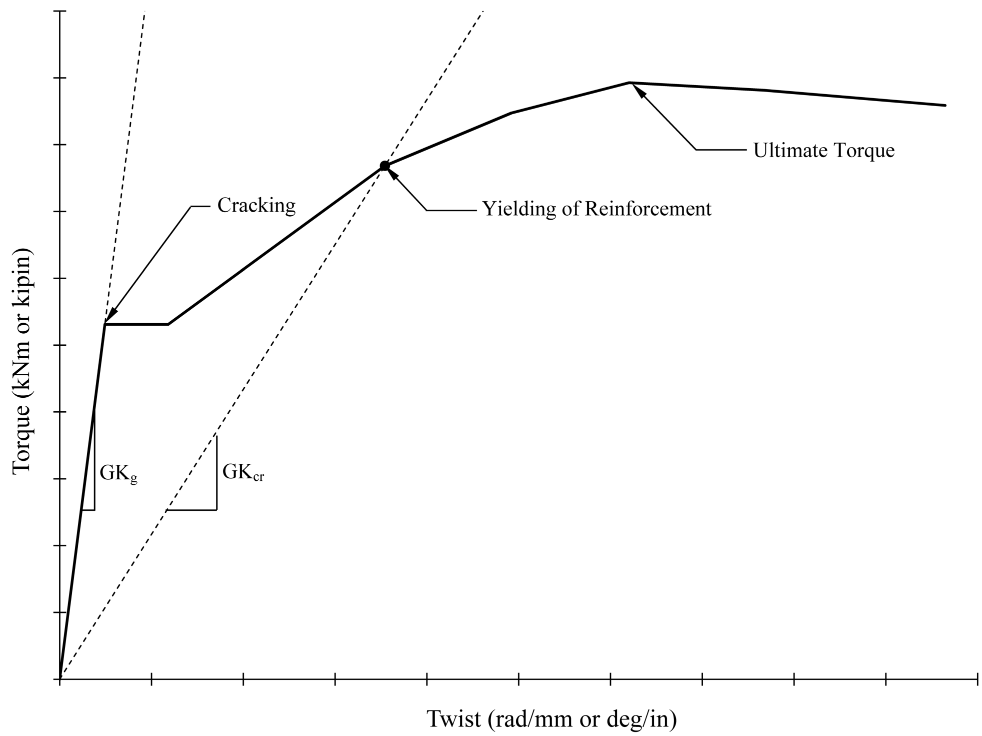


Figure 1 – Typical torque-twist response of a reinforced concrete member subjected to pure torsion

The first expression for the post-cracking stiffness of rectangular reinforced concrete member in pure torsion was developed by Lampert in 1973[7]. Lampert's equation, applicable for both hollow and solid members, calculates GK_{cr} as the following:

$$GK_{cr} = \frac{4E_s A_2^2}{p_2^2\left(\frac{1}{\rho_l}+\frac{1}{\rho_t}\right)} \tag{3}$$

$$\rho_l = \frac{A_l}{A_g} \tag{4}$$

$$\rho_t = \frac{A_t p_h}{A_{cp} s} \tag{5}$$

Where:

- E_s is the Young's modulus of steel
- A_2 is the area enclosed by the centre of corner longitudinal bars
- P_2 is the perimeter of A_2
- A_l is the total area of longitudinal reinforcement in the member
- A_g is the gross area of concrete in the cross section
- A_t is the area of one leg of closed transverse torsional reinforcement
- A_{cp} is the area enclosed by the outside perimeter of the cross section
- s is the spacing of transverse torsional reinforcement

An alternative method for calculating GK_{cr} for general cross sections was formulated by Collins and Mitchell[12] and is shown below:

$$GK_{cr} = \left(\frac{E_s}{2}\right)\frac{4A_o^2}{p_o}\sqrt{\left(\frac{A_t}{s}\right)\frac{A_l + A_p}{p_o}} \tag{6}$$

In this equation, A_o is the area enclosed by the shear flow path, typically taken as $0.85A_{oh}$, where A_{oh} is the area enclosed by the centerline of the outermost closed transverse reinforcement, p_o is the perimeter of the shear flow path, typically taken as $0.9p_h$, where p_h is the perimeter of the centerline of the stirrups, and A_p is the area of prestressed reinforcement.

After cracking, the mechanism for resisting the torsion is considerably more flexible than before, with the residual stiffness typically being significantly reduced to between 5% and 20% of its original value. Equations (3) and (6), as well as other formulations for GK_{cr} in the literature, generally agree that the main determining factors for the post-cracking stiffness are the quantity of reinforcement and the area enclosed by the transverse hoop steel. To illustrate this, Figure 2 shows the relationship between the quantity of reinforcement present and the ratio of GK_{cr} to GK_g, henceforth defined as μ, using Lampert's equation for GK_{cr}. For this particular member, even providing impractically large amounts of reinforcement lead to μ being less than 15%.

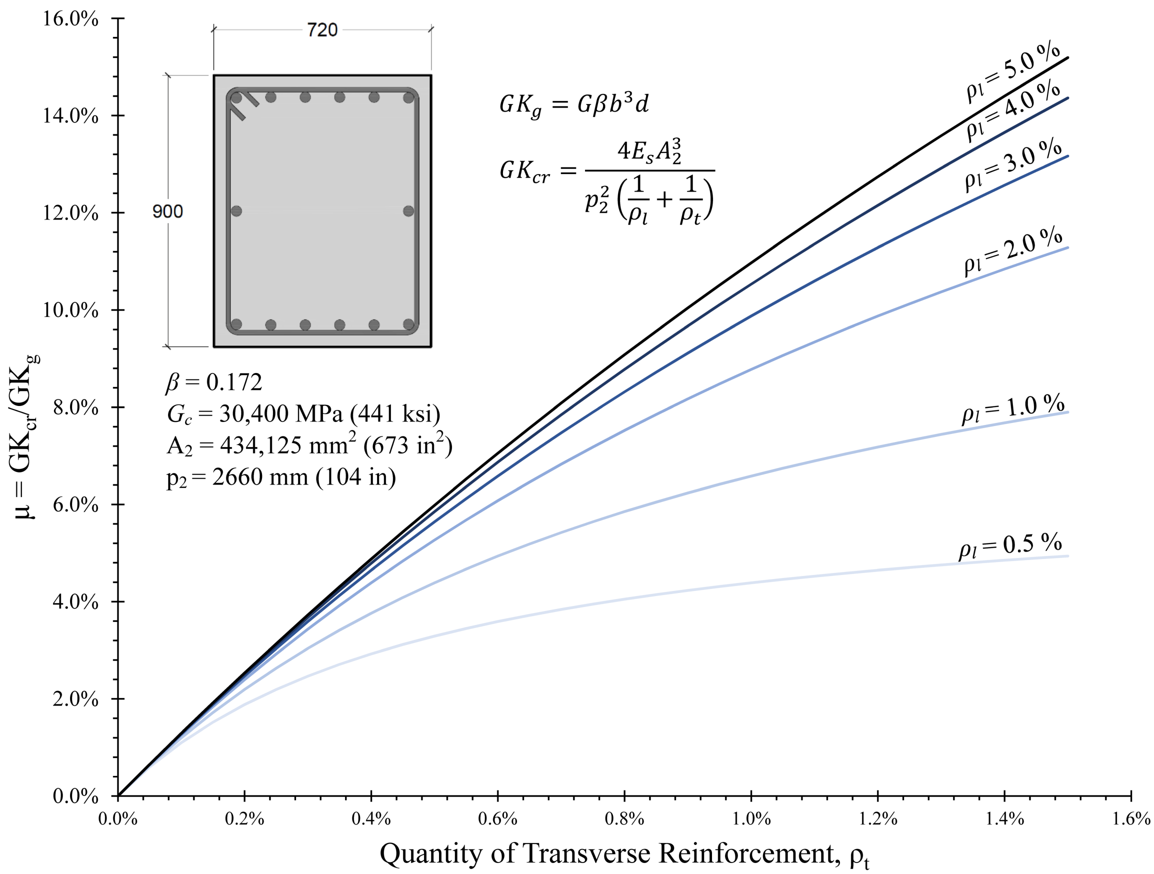


Figure 2 – Influence of reinforcement on μ using Lampert's expression for GK_{cr}

In practical situations, reinforced concrete members are generally subjected to simultaneous moment, shear and torsion, which contrasts with the simpler case of pure torsion. In deriving his equation for GK_{cr} however, Lampert concluded that the torsional stiffness is independent of the presence of the shear force and is only minorly affected by the presence of bending moment[7]. Hence, using the post-cracking stiffness for a member in pure torsion is an appropriate approach even when other actions are also present.

DESIGN PROCEDURE

A design procedure that takes into account torsional stiffness in the context of serviceability is shown in Figure 3. Although the flowchart shows the full solution space with respect to potential design outcomes, the case study and design example in the following section will be centered around the stiffness design path for an indeterminate structure (compatibility torsion), which can be summarized as follows:

STEP 1: *Structural analysis model run with uncracked torsional stiffness values*
- *a. Does the member torsion exceed the cracking torsion?*
  - *i. Yes – Evaluate structure (Proceed to Step 2)*
  - *ii. No – Follow ACI 318-19 design procedure (if the torsion is higher than the threshold value)*

STEP 2: *Is the structure statically determinate or statically indeterminate?*
- *a. Indeterminate – compatibility torsion case (Proceed to Step 3)*
- *b. Determinate – equilibrium torsion case (Proceed to Step 4)*

STEP 3: *Update structural analysis model with ~0 cracked torsional stiffness*
- *a. Are deflections above those required for serviceability?*
  - *i. Yes – Update stiffness (Proceed to Step 4)*
  - *ii. No – Follow ACI 318-19 design procedure*

STEP 4: *Update structural analysis model with maximum cracked torsional stiffness*
- *a. Are deflections above those required for serviceability?*
  - *i. Yes – Serviceability cannot be satisfied by updating stiffness*
  - *ii. No – Update stiffness (Proceed to Step 5)*

STEP 5: *Interpolate to find target stiffness target*

STEP 6: *Design for target stiffness – check to ensure strength design satisfied*

In this procedure, a key parameter to calculate is $\mu_{max,}$ which is defined as the maximum ratio of $\mu = GK_{cr}/GK_g$ that can be achieved given an excessively reinforced section. This term is calculated by selecting appropriately large ratios of longitudinal and transverse steel (defined in Eq. 3), for example: $\rho_l = 4.5\%$ and $\rho_t = 1.5\%$ (chosen at the discretion of the designer). μ_{max} is an upper limit of μ and does not require the designer to design the member first, meaning that it can be determined at the outset of modelling.

In lieu of a more detailed and iterative analysis, the proposed methodology suggests calculating the target torsional stiffness (μ_{target}) from a linear interpolation based on the deflection results from the two models ($\mu \approx 0$ and $\mu = \mu_{max}$) which has been shown to give reasonable results for the case study discussed in this paper.

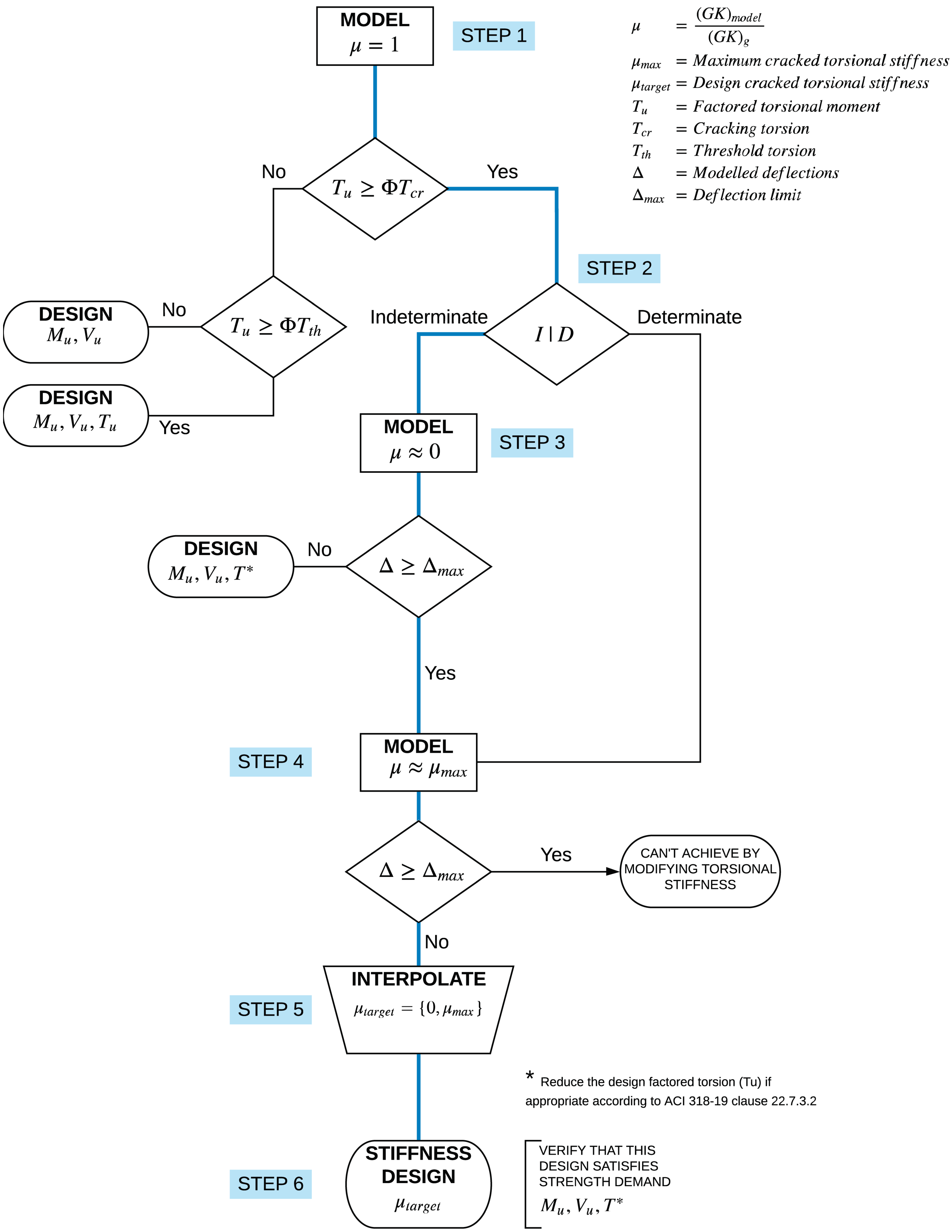


Figure 3 – Torsional design procedure (stiffness method for indeterminate structure highlighted)

TORSIONAL STIFFNESS CASE STUDY

The following case study will introduce the challenges encountered in the design of a series of a reinforced concrete spandrel beams supporting a cantilevered roof; the goal is to demonstrate a situation where post-cracking stiffness in the context of serviceability criteria was an important design consideration. A series of spandrel beams, spanning between columns, together with the roof slab back-span, were supporting a cantilevering beam-slab system that formed the curved overhanging canopy. For further clarification of the structure, see the renderings (Fig. 4) that illustrate the large roof cantilevers. Note that in the final condition, the ribbed-slab cantilevering beyond the façade is hidden by the finishes required to achieve a flat continuous soffit between the interior and exterior portions of the building.

Figure 4– Renderings of case study structure showing roof overhangs (Courtesy of heneghan peng architects / Kearns Mancini Architects, and Luxigon as renderer)

Modelling and design approach

The design of the roof was unusual for the following reasons:

- *Geometry:* long cantilevers, in excess of 5.5m (~18ft) at certain locations.
- *Live Load:* the roof was accessible to the public and was hence designed to carry a full occupancy live load of 4.8kPa (~100psf)
- *Dead Load:* the whole roof was covered in an intensive green roof, with a varying additional load ranging between 1.6-4.0kPa (~33-84psf).
- *Architectural Requirements:* The maximum dimensions of all structural roof members were fixed to satisfy the requirements for installation of insulation and finishes.

The preliminary strength design was based on the simplified approach for an indeterminate structure discussed in the torsional design procedures section – the torsional stiffness of the spandrel beam is reduced to approximately zero and the forces are redistributed and carried in flexure throughout the rest of the system. In a paper discussing the relative distribution of torsion and bending moments between beams and slabs[13], Gouda presents a similar case: a spandrel beam supporting an outer cantilevered beam-slab assembly on one side, and a continuous inner slab on the other. Without applying the simplified approach, the spandrel beam restrains the bending of the cantilevered slab by an unknown amount based on the relative stiffnesses of the members. But if the torsional stiffness is reduced to zero, the beam can be considered as a “knife-edge” support (i.e. rotation unrestrained), and the inner slab becomes the effective back-span to this cantilever and resists the full moment. See Figure 5 for a simplified depiction of the roof assembly in the case study structure, and the 2D free body diagrams for both modelling cases.

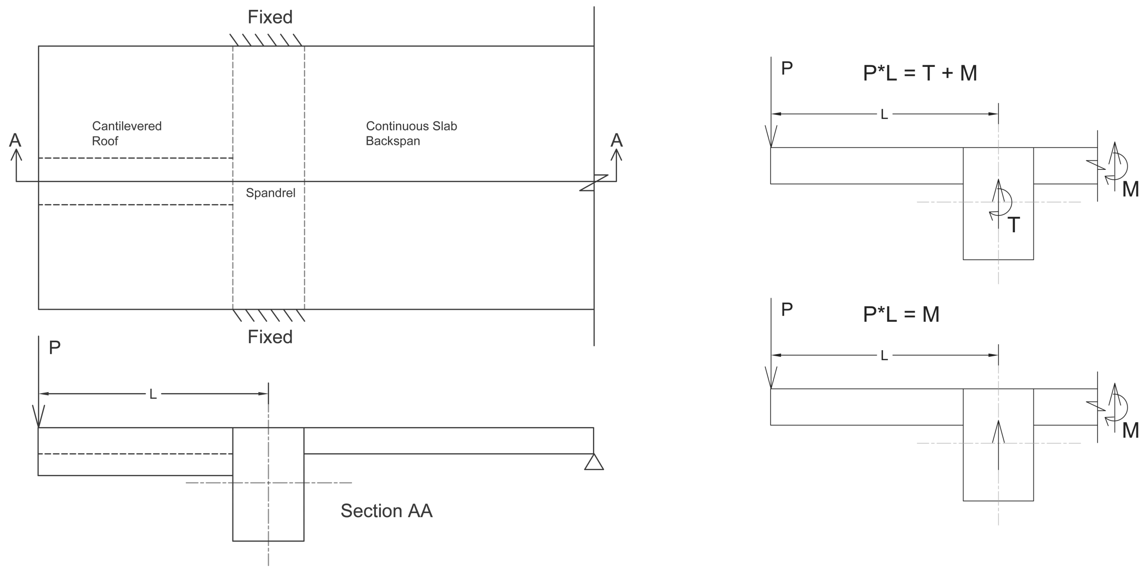
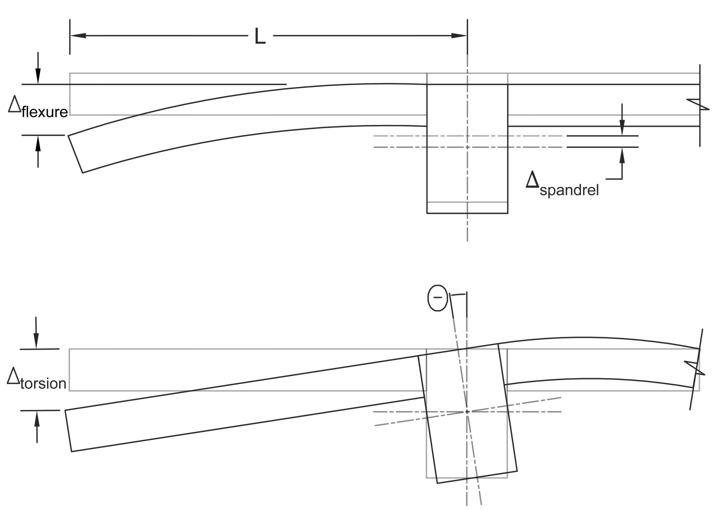


Figure 5 – Simplified cantilevered roof structure (left), and resulting moments in the non-simplified, and simplified cases (right)

Cantilever deflections

The preliminary strength design was performed satisfactorily, but when assessing the live load deflections, it was found that the ends of the cantilevers exceeded the immediate live-load and long-term deflections limits that were specified for the finishes. In this situation, the total tip deflection is a function of the cantilever beam flexural deflection and the spandrel beam twist, as illustrated in Fig. 6. Based on how the finishes were attached, the relative deflections were measured from the position of the deformed spandrel, hence its flexural deformations could be ignored. Aside from minor changes in slab tributary load, the flexural deformation of the cantilever can be considered independent of the torsional stiffness of the spandrel. From the particular mechanics and geometry of the structure under discussion, it was found that the deformations due to the twist of the beam were approximately double those due to flexure. Therefore, increasing the torsional stiffness to minimize this twist was considered a valid approach to take. To illustrate the stiffness design approach, only the immediate live-load case will be discussed – the deflection issues were magnified in the long-term case, but the same procedure can be applied to an analysis in any time domain.



$$\Delta_{Flexure} = \frac{wL^4}{8EI}$$

$$\Delta_{Torsion} = \sin \Theta \cdot L$$

Figure 6 – Cantilever roof tip deflection as a function of flexural and torsional deformation (not to scale)

DESIGN EXAMPLE

The following section presents the design of the spandrel beam (Fig. 7) based on the procedure that was outlined in the previous section. A summary of the design forces at the critical section, and the immediate cantilever tip deflections under live loads from the analysis model at each design step (Fig. 3) are shown in Table 2. Note that although Lampert's equation is used in this design example in steps 4 and 6 to calculate GK_{cr}, any secant-based method may be used.

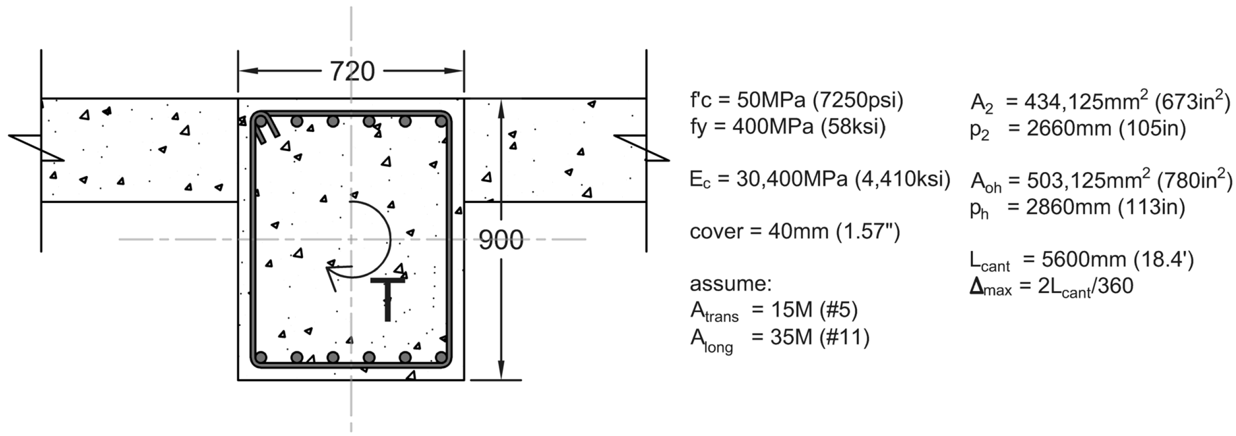


Figure 7 – Spandrel beam parameters

Table 2. Summary of analysis results

Step #	μ	Strength Design			Serviceability
		M_u^- kN•m (kip-ft)	V_u kN (kip)	T_u kN•m (kip-ft)	Δ_{total} mm (in)
1	1	868 (640)	751 (169)	414 (305)	27.6 (1.09)
3	≈0	948 (699)	796 (179)	≈0	32.4 (1.28)
4	0.148	893 (659)	762 (179)	202 (149)	30.2 (1.19)

Step 1

See Table 2 for the design forces at the critical section, and the immediate cantilever tip deflections under live loads from the analysis model with uncracked torsional stiffness ($\mu = 1$).

$\phi T_{cr} = 228$ kN•m (168 kip-ft). Since $T_u > \phi T_{cr}$ the section is cracked.

Step 2

The cantilever roof is supported by both the spandrel beam and the back-span slab. As this is a statically indeterminate system, the redistribution of moments is possible, hence the compatibility design procedure should be followed. T_u is reduced to ϕT_{cr} per clause 22.7.3.2.

Step 3

See Table 2 for the design forces at the critical section, and the immediate cantilever tip deflections under live loads from the analysis model with no torsional stiffness ($\mu \approx 0$).

Δ_{max}= 31.1mm (1.22in), calculated for a cantilever of length 5600mm. Since $\Delta > \Delta_{max}$ the stiffness should be modified.

Step 4

The maximum cracked torsional stiffness was found to be 14.8% ($\mu_{max} \approx 0.148$), based on assumed maximum reinforcement quantities of ρ_l =4.5% and ρ_t =1.5%.

$$GK_g = (0.4E_c) \cdot 0.172 \cdot (900 \cdot 720^3) = 702.6 \times 10^3 \text{ kNm}^2 \text{ } (245.1 \times 10^6 \text{ kip} - \text{in}^2)$$

$$GK_{cr,max} = \frac{4 \cdot 200{,}000 \cdot 434{,}125^3}{2660^2 \cdot \left(\frac{1}{0.045} + \frac{1}{0.015}\right)} = 104.1 \times 10^3 \text{ kNm}^2 (36.3 \times 10^6 \text{ kip} - \text{in}^2)$$

See Table 2 for the design forces at the critical section, and the immediate cantilever tip deflections under live loads from the analysis model with no maximum cracked stiffness ($\mu_{max} \approx 0.148$).

Since $\Delta < \Delta_{max}$ the deflection limit is attainable with a torsional stiffness below the maximum.

Step 5

The target stiffness to achieve the desired deflection is μ = 8.75%.

$$\mu_{target} = \frac{\left(\Delta_{\mu_{target}} - \Delta_{\mu_0}\right)}{\left(\Delta_{\mu_{max}} - \Delta_{\mu_0}\right)} (\mu_{max} - \mu_0) \tag{7}$$

$$\mu_{target} = \frac{(31.1 - 32.4)}{(30.2 - 32.4)} (0.148 - 0) = 0.0875$$

Step 6

Based on Fig. 2, it is anticipated that a large quantity of reinforcement will be necessary to obtain μ = 8.75%. Providing 16-35M bars as longitudinal reinforcement (6 on the top and 6 on the bottom, with 4 at mid-height) yields ρ_l = 2.47%. The corresponding quantity of transverse reinforcement to obtain the target value of μ, ρ_t, was found to be 0.909%, which corresponds to 15M stirrups spaced at 97.1 mm (3.82 in).

$$GK_{cr} = \mu_{target} \cdot GK_g = (0.0875) \cdot (702.6 \times 10^{12}) = 61.5 \times 10^3 \text{ kNm}^2 \text{ } (21.5 \times 10^6 \text{ kip} - \text{in}^2)$$

$$\rho_t = \left(\frac{4E_s A_2^3}{p_2^2 (GK)_{cr}} - \frac{1}{\rho_l}\right)^{-1} = \left(\frac{4 \cdot 200{,}000 \cdot 434{,}125^3}{2660^2 \cdot 61.5 \times 10^{12}} - \frac{1}{0.0247}\right)^{-1} = 0.00909$$

$$s = \frac{A_t p_h}{A_{cp} \rho_t} = \frac{200 \cdot 2860}{648{,}000 \cdot 0.00909} = 97.1 \text{ mm (3.82 in)}$$

After designing for the required stiffness, the member should then be checked using the provisions in ACI318-19 to ensure that it can carry the required loads. The section size should first be checked to avoid crushing or excessive cracking according to clause 22.7.7.1:

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7 A_{oh}^2}\right)^2} \leq \phi 10 \sqrt{f_c'} \tag{8}$$

$$\sqrt{\left(\frac{796}{720 \cdot 827.5}\right)^2 + \left(\frac{228 \cdot 2860}{1.7 \cdot 503{,}125^2}\right)^2} = 2.02 \text{ MPa (296 psi)} < 0.75 \cdot 10 \cdot \sqrt{50} = 4.40 \text{ MPa (638 psi)}$$

The torsional strength now needs to be calculated as the minimum of the torsion causing yielding of the longitudinal or transverse reinforcement. The strength associated with yielding of the longitudinal steel, $T_{n,long\ yield}$, can be directly evaluated by an equation presented in another paper in this volume[14]. If it is assumed that the flexural lever arm jd is equal to 0.9d, the ratio $\omega = M_n/T_n = 4.145$ and the yield strength of steel is $f_y = 400$ MPa (58 ksi):

$$T_{n,long\ yield} = \frac{4A_o A_s f_y jd \tan 45°}{p_h jd + 4A_o \omega \tan 45°} \tag{9}$$

$$T_{n,long\ yield} = \frac{4 \cdot 0.85 \cdot 503{,}125 \cdot (8 \cdot 1000) \cdot 400 \cdot 0.9 \cdot 827.5 \cdot 1}{2860 \cdot 0.9 \cdot 827.5 + 4 \cdot 0.85 \cdot 503{,}125 \cdot 4.145 \cdot 1} = 442 \text{ kNm (326 kip} - \text{ft)}$$

The torsional strength associated with yielding of the stirrups, $T_{n,trans\ yield}$, can similarly be directly evaluated using another equation presented in the same paper[14]. Using the ratio $\xi = V_n/T_n = 0.003491$ mm^{-1} and calculating V_c in the absence of axial load using clause 22.5.5.1:

$$V_c = 2\sqrt{f_c'} b_w d \tag{10}$$

$$V_c = 2 \cdot \sqrt{50} \cdot 720 \cdot 827.5 = 699 \text{ kN (157 kips)}$$

$$T_{n,trans\ yield} = \frac{2A_o\left(sV_c + 2A_t d f_y\right)}{s(2A_o \xi + 2d \tan 45°)} \tag{11}$$

$$T_{n,trans\ yield} = \frac{2 \cdot 0.85 \cdot 503{,}125 \cdot (97.1 \cdot 699 + 2 \cdot 200 \cdot 827.5 \cdot 400)}{97.1(2 \cdot 0.85 \cdot 503{,}125 \cdot 0.003491 + 2 \cdot 827.5 \cdot 1)} = 380 \text{ kNm (280 kip} - \text{ft)}$$

$T_{n,trans\ yield}$ must also be limited by the strength of the member in pure torsion, $T_{n,pt}$, which is calculated using clause 22.7.6.1.a:

$$T_{n,pt} = 2A_o \frac{A_t f_{yt}}{s} \cot 45° \tag{12}$$

$$T_{n,pt} = 2 \cdot 0.85 \cdot 503{,}125 \frac{200 \cdot 400}{97.1} \cdot 1 = 705 \text{ kNm (520 kip} - \text{ft)}$$

The factored torsional strength, ϕT_n, is then the minimum of $T_{n,long\ yield}$, $T_{n,trans\ yield}$ and $T_{n,pt}$:

$$\phi T_n = \phi \min\{T_{n,long\ yield}, T_{n,trans\ yield}, T_{n,pt}\} \tag{13}$$

$$\phi T_n = 0.75 \cdot \min\{442, 380, 705\} = 285 \text{ kNm((210 kip} - \text{ft)} > T_u = 228 \text{ kNm (168 kip} - \text{ft)}$$

The factored shear strength is then directly obtained as a function of ϕT_n and ξ:

$$\phi V_n = \phi \xi T_n = 0.75 \cdot 0.003491 \cdot 380 = 995 \text{ kN} > \text{V}_\text{u} = 796 \text{ kN (179 kips)}$$

Summary

The resulting design to achieve the target stiffness is generally conservative, with the shear and torsional strengths both being approximately 25% higher than the factored shear and torsion demand. Note that increasing the post-cracking torsional stiffness requires a considerable amount of both longitudinal and transverse reinforcement. The calculated short-term deflection was decreased by 2.2mm, about 7%, which is within the range of inherent variability in structural deflections. Therefore, in this situation it might be deemed more economical to attempt to reduce deflections by employing other strategies based on modifying the surrounding structure. Ultimately, the decision on what strategy to use is at the discretion of the design engineer; the method proposed in this paper being one of many tools available.

CONCLUSION

In the analysis of reinforced concrete structures, it is important to model the post-cracking stiffness of individual members as accurately as possible, as an incorrect assignment can lead to either unconservative or unrealistic design loads. For example, in a statically indeterminate structure, the relative stiffness of a member is proportional to its resultant internal forces and moments, hence an unusually high or low stiffness will lead to incorrect values. Sectional forces are also generated in elements as a function of the deformation they undergo to maintain compatibility with the surrounding structure. In such situations, assigning an overly high stiffness to an element will result in unrealistic sectional forces and moments, which would in actuality have been redistributed upon cracking. Furthermore, in situations where displacements are strongly related to the rotation of twisting members, having an accurate estimate of the post-cracking stiffness is necessary to check displacement limits under service loads. Knowing the post-cracking torsional stiffness is hence important when designing for both ultimate and serviceability limits states.

Post-cracking stiffness is a function of the amount of reinforcement in a section; therefore, calculating an accurate value becomes an iterative process as the reinforcement is not known at the outset of a design. To simplify this process, ACI 318-19 provides typical values that can be used for post-cracking flexural stiffness as a function of the gross cross-section, in lieu of a more accurate calculation. Yet there is no equivalent code guidance with respect to post-cracking torsional stiffness. This paper has aggregated the academic literature on the topic and used this information as the basis for a stiffness-based design procedure. The procedure is then implemented in a design case study of a reinforced concrete spandrel beam, which was required to reach a certain level of torsional stiffness to satisfy deflection requirements. The ability to directly design a beam for a certain stiffness removes the need to iteratively model and design a structure when an accurate measure of torsional stiffness and moment redistribution is required. The method is broadly formulated to be used with different material models and can be applied to any situation where a specific value of post-cracking torsional stiffness is required by a structural analysis.

NOTATION

A_2	=	Area enclosed by the centerlines of the corner longitudinal bars
A_{cp}	=	Area enclosed by outside perimeter of concrete cross section
A_g	=	Gross area of concrete in the cross section
A_l	=	Total area of longitudinal reinforcement to resist torsion
A_o	=	Area enclosed by shear flow path
A_{oh}	=	Area enclosed by centerline of closed transverse reinforcement
A_p	=	Area of prestressed reinforcement
A_s	=	Area of longitudinal reinforcement on the flexural tension side of member
A_s'	=	Area of longitudinal reinforcement on the flexural compression side of member
A_t	=	Area of one leg of closed transverse torsion reinforcement
b	=	Short dimension of a rectangular cross section
b_w	=	Effective web width within depth d
d	=	Distance from the extreme compression fibre to centroid of longitudinal tension reinforcement
E_c	=	Young's modulus of concrete
E_s	=	Young's modulus of reinforcing steel
f_c'	=	Specified compressive strength of concrete
f_y	=	Specified yield strength of reinforcement
G	=	Shear modulus
GK_g	=	Uncracked torsional stiffness
GK_{cr}	=	Cracked torsional stiffness equal to the torque causing yielding divided by the twist at yield
h	=	Long dimension of a rectangular cross section
jd	=	Flexural lever arm
K	=	St. Venant torsion constant
M_u	=	Moment demand at section
p_2	=	Perimeter enclosed by A_2
p_h	=	Perimeter of the centerline of the closed transverse torsion reinforcement
p_o	=	Perimeter of the shear flow path
s	=	Spacing of transverse reinforcement
T_{cr}	=	Cracking torque
T_n	=	Nominal torsional strength of the member
T_u	=	Torsion demand at section

V_c = Shear strength attributed to concrete
V_u = Shear demand at section
B = Coefficient for determining K for a rectangular cross section
μ = Ratio of cracked torsional stiffness to uncracked torsional stiffness, GK_{cr}/GK_g
μ_{max} = Upper limit of μ based on maximum values of ρ_l and ρ_t
ξ = Ratio of shear and torsion strength, V_n/T_n
ρ_l = Quantity of longitudinal reinforcement
ρ_t = Quantity of transverse reinforcement
ϕ = Reduction for shear and torsion in ACI 318-19. ϕ =0.75.
ψ = Twist of member in units of angle per unit length
ω = Ratio of moment and torsional strength, M_n/T_n

REFERENCES

[1] ACI Committee 318 "Building Code Requirements for Structural Concrete (ACI 318-19) and Commentary (ACI 318R-19)," American Concrete Institute, Farmington Hills, MI, 2019, 623 pp.

[2] Fisher, G., (editor), "SP-018: Torsion of Structural Concrete," ACI Symposium Publication, Vol. 18, 1968. 512 pp.

[3] ACI Committee 318 "Building Code Requirements for Reinforced Concrete (ACI 318-71)," American Concrete Institute, Detroit, MI, 1971, 144 pp.

[4] Fisher, G., Szilard, R., Zia, P., (editors), "SP-035: Analysis of Structural Systems for Torsion," American Concrete Institute Special Publication, Vol. 35, 1973. 438 pp.

[5] Collins, M. P. and Lampert, P., "Redistribution of Moments at Cracking - The Key to Simpler Torsion Design?," *ACI Special Publication*, Vol. 35, 1973, 343-383.

[6] Mitchell, D. and Collins M. P., "Detailing for Torsion," *ACI Symposium Publication*, vol. 73, no. 9, pp. 506–511, Sep. 1976.

[7] Lampert, P., "Postcracking Stiffness of Reinforced Concrete Beams in Torsion and Bending," *ACI Special Publication*, vol. 35, pp. 343–433, 1973.

[8] Hsu, T. T. C., "Post-Cracking Torsional Rigidity of Reinforced Concrete Sections," *ACI Journal Proceedings*, vol. 70, no. 5, 1973.

[9] Bruun, E.P.G., "The Hybrid Panel-Truss Element: Developing a Novel Finite Element for the Nonlinear Analysis of Reinforced Concrete Beams and Shells," Master's, University of Toronto, Canada, 2017.

[10] Bruun, E.P.G. and Bentz, E.C., "Experimental Procedures for Displacement-Controlled Pure Torsion Tests on Reinforced Concrete Shells," in *7th International Conference on Advances in Experimental Structural Engineering*, Pavia, Italy, 2017, p. 21.

[11] Teng, T. and Teng, S., "Effective Torsional Rigidity of Reinforced Concrete Members," *ACI Structural Journal*, vol. 101, no. 2, 2004.

[12] Collins, M. P. and Mitchell, D., *Prestressed Concrete Structures*. Toronto, Canada: Response Publications, 1997.

[13] Gouda, M. A., "Distribution of Torsion and Bending Moments in Connected Beams and Slabs," *Journal of the American Concrete Institute*, vol 56, no. 2, p. 18, 1960.

[14] Kuan A., Bruun, E.P.G., Bentz, E.C., Collins, M.P., "Alternative Design Procedures for Torsion in ACI 318-19: A Comparative Study," *ACI Symposium Publication*, vol. 344, pp. 65-92, 2020.